Int. J. Heat Mass Transfer. Vol. 32, No. 5, pp. 935–941, 1989
Printed in Great Britain

0017-9310/89 \$3.00+0.00

Sinusoidal perturbation solutions for planar solidification

NAI-YI LI and J. R. BARBER

Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

(*Received* 25 *March* 1988 *and in final form* 18 *October* 1988)

Abstract—A linear perturbation method is used to solve the two-dimensional heat conduction problem in which a liquid, initially at the melting temperature, becomes solidified by heat transfer to a plane mold the temperature of which is approximately uniform, but contains a small sinusoidal perturbation in one space dimension. Results are obtained for the consequent sinusoidal perturbation in the melt/solid boundary as a function of time and for the temperature distribution throughout the solid shell. These results are expressed in terms of a series of confluent hypergeometric functions which shows good convergence for practical values of material properties and temperatures. The inverse problem, in which the melt/solid boundary is prescribed and the mold temperature is to be determined, is also briefly discussed.

1. INTRODUCTION

THE PROBLEM of heat conduction in a body with melting or solidification—also known as the Stefan problem [1]—is of considerable practical importance, because of applications to the processes of casting, welding and the formation of ice.

The motion of the boundary between the solid and liquid phases makes the problem inherently nonlinear and hence only the simplest boundary value problems can be treated by analytical methods [2, 3]. Neumann's solution [4] to the one-dimensional problem of solidification in a semi-infinite region remains one of the few exact solutions available. Approximate analytical solutions have been obtained by a variety of techniques, such as the use of the heat balance [5], variational [6], and embedding methods [7]. Also, many problems have been investigated by numerical methods, notably the finite difference [8, 9] and finite element methods [10, 11].

In the casting process, solidification is effected by the transfer of heat from the melt through the mold to the surroundings. Thus, solidification starts at the mold surface and a shell of solidified material grows from this surface towards the center of the casting. The growth of this shell is often found to be non-uniform [12], even in nominally one-dimensional solidification, in which case residual stresses will be developed tending to produce waviness in the nominally plane cast surface. This non-uniform shell growth could be a response to a pre-existing nonuniformity in mold temperature, but Richmond and Huang [13] have noted that the mold temperature itself is also influenced by the contact conditions between the shell and the mold. Thus, it is possible that a small spatial perturbation in mold temperature could be unstable, due to thermomechanical coupling. Instabilities of this kind have already been reported for the one-dimensional conduction of heat across an interface between two thermoelastic half-spaces [14].

In this paper, we investigate heat conduction problems in which one-dimensional solidification is perturbed by the superposition of a small spatial sinusoidal variation in temperature parallel to the plane of the mold surface. In particular, we shall obtain solutions for the temperature field and the advance of the solid/melt boundary for the case where the mold temperature varies sinusoidally with position. We also briefly discuss the inverse problem, in which the solid/melt boundary has a prescribed sinusoidal shape as a function of time and the corresponding mold temperature is to be found. The solution is developed by linear perturbation of Neumann's solution [4] for planar solidification. Perturbation methods have previously been applied to the solidification problem by Pedroso and Domoto [15] and Mullins and Sekerka [16].

2. MATHEMATICAL FORMULATION

We consider the two-dimensional problem of a liquid initially at its melting temperature T_m, in the half-space $y > 0$, in perfect thermal contact with a plane mold at $y = 0$, the temperature of which $F(x, t)$ $(< T_m)$ is a prescribed function of one space coordinate and time, t. After a time t, the liquid solidified near the mold forms a solid shell of thickness $h(x, t)$. Thus, $h(x, t)$ defines the moving interface between the solid and liquid phases as shown in Fig. 1. We assume that the density ρ, thermal diffusivity k, conductivity K, and specific heat c of the solid phase are constant, independent of temperature and time.

The temperature of the solid $T(x, y, t)$ must satisfy the heat conduction equation

NOMENCLATURE

A, a, b	arbitrary constants
$A_n, \ldots, E_n$	coefficients in series solution
c	specific heat
$F(x, t)$	mold temperature
$h(x, t)$	thickness of solidified material
H	dimensionless thickness of solidified material, (mh)
HT	see equation (46)
k	thermal diffusivity
K	conductivity
L	latent heat of fusion
m	parameter of sinusoidal perturbation
p	see equation (33)
s	integral transform parameter
t	time
$T(x, y, t)$	temperature
T_m	melting temperature
ΔT_0	mean temperature difference between mold and liquid, $(T_m - T_0^*)$
W	see equation (38)
x, y	spatial coordinates
Y	dimensionless coordinate, $y/\sqrt{(4kt)}$.

Greek symbols

α	separation constant, see equation (27)
ε_n	coefficients for H_1 in series expansion form
λ	material constant in Neumann's solution defined by equation (7)
Λ	material constant, (L/c)
ρ	density
τ	dimensionless time, (m^2kt)
Φ	confluent hypergeometric function.

Subscripts

0	zeroth-order term
1	first-order term.

Superscript

*	coefficients of prescribed mold temperature.

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t} \tag{1}$$

and the boundary conditions

$$T(x, h, t) = T_m \tag{2}$$

$$K\frac{\partial T}{\partial y}(x, h, t) = L\rho\frac{\partial h}{\partial t}(x, t) \tag{3}$$

$$T(x, 0, t) = F(x, t) \tag{4}$$

where L is the latent heat of fusion of the material.

Equation (2) states that there must be continuity of temperature at the solid/melt interface, while equation (3) defines an energy balance between the heat conducted away from the interface into the solid and the latent heat released during solidification. The temperature of the solid/mold interface is prescribed by equation (4).

In the special case where the $F(x, t)$ is a constant $(= T_0^*)$, the temperature becomes independent of x and the problem reduces to Neumann's problem, with the well-known solution [4]

$$T_0(y, t) = \frac{\Delta T_0}{\operatorname{erf}\lambda}\operatorname{erf}\left(\frac{y}{\sqrt{(4kt)}}\right) + T_0^* \tag{5}$$

$$h_0(t) = \lambda\sqrt{(4kt)} \tag{6}$$

where $\Delta T_0 \equiv T_m - T_0^*$ and λ is a dimensionless constant determined from the equation

$$\lambda e^{\lambda^2}\operatorname{erf}\lambda = \frac{c\Delta T_0}{L\sqrt{\pi}} \tag{7}$$

which is derived from boundary condition (3). For the case of the freezing of water by conduction of heat to the surroundings a few degrees below the freezing point, λ is small. However, for materials with high melting points, such as metals and rocks cooled by contact with a surface near room temperature, λ is typically in the range $0.5 < \lambda < 1$ [4].

We now consider the case in which the prescribed mold temperature is of the form

$$F(x, t) = T_0^* + T_1^*(t)\cos(mx) \tag{8}$$

where the second term defines a small sinusoidal perturbation on the constant temperature, T_0^*. Notice that the perturbation must be small in comparison with the temperature *difference* between the melt and the mold, i.e. $T_1^*(t) \ll \Delta T_0$.

We anticipate a corresponding small perturbation in the temperature field and the thickness of the solidified layer, which can be expressed through the equations

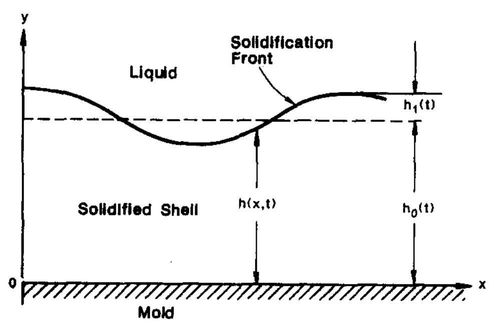

FIG. 1. Geometry of the system.

$$T(x,y,t) = T_0(y,t)+T_1(y,t)\cos(mx);$$

$$T_0(y,t)-T_0^* \gg T_1(y,t) \quad (9)$$

$$h(x,t) = h_0(t)+h_1(t)\cos(mx); \quad h_0(t) \gg h_1(t). \quad (10)$$

Substituting these equations into equation (1) and separating periodic and uniform terms in x, we obtain

$$\frac{\partial^2 T_0}{\partial y^2} = \frac{1}{k}\frac{\partial T_0}{\partial t} \quad (11)$$

$$\frac{\partial^2 T_1}{\partial y^2} - m^2 T_1 = \frac{1}{k}\frac{\partial T_1}{\partial t}. \quad (12)$$

Since the perturbation is small, we can expand the temperature field in the vicinity of the mean solid/melt interface position, $y = h_0(t)$, in the form of a Taylor series, in which case boundary condition (2) can be written as

$$T_0(h_0,t)+\frac{\partial T_0(h_0,t)}{\partial y}h_1(t)\cos(mx) + \frac{\partial^2 T_0(h_0,t)}{\partial y^2}\frac{h_1^2(t)\cos^2(mx)}{2!}+\cdots+\left[T_1(h_0,t) + \frac{\partial T_1(h_0,t)}{\partial y}h_1(t)\cos(mx)+\cdots\right]\cos(mx) = T_m. \quad (13)$$

Separating periodic and uniform terms and dropping second-order and product terms in the small quantities, T_1, h_1, we obtain the two first-order equations

$$T_0(h_0,t) = T_m \quad (14)$$

$$h_1(t)\frac{\partial T_0(h_0,t)}{\partial y}+T_1(h_0,t) = 0. \quad (15)$$

A similar procedure applied to boundary condition (3) also yields the two equations

$$K\frac{\partial T_0(h_0,t)}{\partial y} = L\rho\frac{\mathrm{d}h_0(t)}{\mathrm{d}t} \quad (16)$$

$$K\left[\frac{\partial T_1(h_0,t)}{\partial y}+h_1(t)\frac{\partial^2 T_0(h_0,t)}{\partial y^2}\right] = L\rho\frac{\mathrm{d}h_1(t)}{\mathrm{d}t}. \quad (17)$$

Finally, the solid/mold boundary condition (4), with equation (8), gives the two equations

$$T_0(0,t) = T_0^* \quad (18)$$

$$T_1(0,t) = T_1^*(t). \quad (19)$$

Governing equation (11) with boundary conditions (14), (16), and (18) are clearly identical to those of the classical Neumann problem and hence T_0, h_0 are given by equations (5)–(7). The remaining equations (12), (15), (17), and (19) then define a corresponding problem for the perturbed quantities, T_1, h_1.

We note that if the expansion of equation (13) had been taken to include second-order small quantities, equation (14) would include a term involving T_1 and it would also be necessary to introduce a further perturbation term varying with cos $(2mx)$. Thus, for larger perturbations, the mean advance of the solid/melt interface deviates from the Neumann solution and the perturbed shape of the solidified shell ceases to be sinusoidal.

2.1. *Dimensionless formulation*

Before proceeding to the solution for T_1, h_1, it is convenient to introduce the dimensionless variables

$$Y = \frac{y}{\sqrt{(4kt)}}$$

$$\tau = m^2 kt$$

$$H(\tau, mx) = mh(x,t) = H_0(\tau)+H_1(\tau)\cos(mx). \quad (20)$$

Governing equation (12) for $T_1(Y,\tau)$ then becomes

$$\frac{1}{4\tau}\frac{\partial^2 T_1}{\partial Y^2} - T_1 = \frac{\partial T_1}{\partial \tau} - \frac{Y}{2\tau}\frac{\partial T_1}{\partial Y} \quad (21)$$

and boundary conditions (15), (17), and (19) become

$$T_1(\lambda,\tau)+\Lambda\frac{\lambda}{\sqrt{\tau}}H_1(\tau) = 0 \quad (22)$$

$$\frac{\partial T_1}{\partial Y}(\lambda,\tau) - 2\Lambda\sqrt{\tau}\frac{\mathrm{d}H_1(\tau)}{\mathrm{d}\tau} - 2\Lambda\lambda^2\frac{H_1(\tau)}{\sqrt{\tau}} = 0 \quad (23)$$

$$T_1(0,\tau) = T_1^*(\tau) \quad (24)$$

respectively, where $\Lambda = L/c$ is a material constant with the dimension of temperature and the terms involving T_0 in equations (15) and (17) have been simplified using equations (5)–(7).

We can eliminate $H_1(\tau)$ in equations (22) and (23) to obtain

$$\frac{\partial T_1}{\partial Y}(\lambda,\tau)+(1/\lambda+2\lambda)T_1(\lambda,\tau) + \frac{2\tau}{\lambda}\frac{\partial T_1}{\partial \tau}(\lambda,\tau) = 0. \quad (25)$$

Thus, the problem is reduced to the determination of a function $T_1(Y,\tau)$ which satisfies equation (21) and boundary conditions (24) and (25), after which the unknown profile of the solid/melt interface can be recovered from equation (22).

We first seek particular solutions to equation (21) in the separated variable form $T_1(Y,\tau) = f(Y)g(\tau)$. With some rearrangement, equation (21) can then be written as

$$\frac{f''+2Yf'}{f} = \frac{4\tau(g'+g)}{g} \quad (26)$$

where a prime denotes the derivative with respect to the appropriate argument. Since the right- and left-hand sides of equation (26) have different independent variables, both sides must be equal to a constant α. Hence we have two ordinary differential equations

$$g' = g(\alpha/4\tau - 1) \quad (27)$$

$$f''+2Yf'-\alpha f=0 \quad (28)$$

with solutions

$$g(\tau)=A\,e^{-\tau}\tau^{\alpha/4} \quad (29)$$

$$f(Y)=a\Phi(-\alpha/4,1/2;-Y^2)+bY\Phi(1/2-\alpha/4,3/2;-Y^2) \quad (30)$$

$$=e^{-Y^2}[a\Phi(1/2+\alpha/4,1/2;Y^2)+bY\Phi(1+\alpha/4,3/2;Y^2)] \quad (31)$$

where Φ is the confluent hypergeometric function [17] and A, a, b are arbitrary constants.

A more general solution can now be obtained by superposing solutions of this form, treating α as a parameter. For example, writing $n=\alpha/4$, we can define a solution to equation (21) in the form

$$T_1(Y,\tau)=e^{-(\tau+Y^2)}\sum_{n=0}^{\infty}\tau^n[A_n\Phi(1/2+n,1/2;Y^2)+B_nY\Phi(1+n,3/2;Y^2)] \quad (32)$$

where A_n and B_n are unknown constants to be determined from boundary conditions (24) and (25). We note that the first hypergeometric function in equation (32) can be written in terms of an Hermite polynomial using [17], 13.6.17.

2.2. *Direct problem—prescribed* $T_1^*(\tau)$

We consider the case where the perturbed temperature at the solid/mold interface varies exponentially in time and sinusoidally in space, i.e.

$$T_1^*(\tau)=F_1\,e^{p\tau} \quad (33)$$

where F_1 is a constant with the dimensions of temperature and p a dimensionless constant. The particular case in which T_1^* is independent of time can be recovered by setting $p=0$. Equations (24), (25), and (32) then yield

$$A_n=\frac{F_1(1+p)^n}{n!} \quad (34)$$

$$B_n=\frac{F_1(1+p)^n}{n!}C_n \quad (35)$$

for constants A_n, B_n, where the new constants C_n are defined through the recurrence relation

$$C_n=[2nW(n-1/2,1/2)+2n\lambda W(n,3/2)C_{n-1}-(1+p)(2n+1)W(n+3/2,1/2)]/[2(1+p)(n+1)\lambda W(n+2,3/2)] \quad (36)$$

and

$$C_0=-\frac{(2\lambda+1/\lambda)}{e^{-\lambda^2}+\frac{\sqrt{\pi}}{2}(2\lambda+1/\lambda)\operatorname{erf}\lambda} \quad (37)$$

$$W(a,b)=\Phi(a,b,\lambda^2). \quad (38)$$

The amplitude, $H_1(\tau)$, of the sinusoidal perturbation of the solid/melt interface can then be found as a function of time from equations (22) and (32) in the form

$$H_1(\tau)=-\frac{\sqrt{\tau}\,e^{-(\tau+\lambda^2)}}{\Lambda\lambda}\sum_{n=0}^{\infty}\tau^n[A_n\Phi(1/2+n,1/2;\lambda^2)+B_n\lambda\Phi(1+n,3/2;\lambda^2)]. \quad (39)$$

It can be shown from the asymptotic properties of the hypergeometric function and equations (33)–(38) that the series in equation (39) (and hence that in equation (32)) is absolutely convergent for all λ, p, and τ. We also note that the method of this section can be employed for more general functions $T_1^*(\tau)$, provided they permit a convergent power series expansion in the range under consideration.

2.3. *Inverse problem—prescribed* $H_1(\tau)$

In certain cases, we may have more information about the advance of the melt/solid boundary than about the temperature history of the mold surface. Such information might be obtained for example by examination of the crystal structure of the finished casting or from observations of the solid shells from an interrupted casting process. It is therefore of interest to consider the case in which the function $H_1(\tau)$ is prescribed and the perturbation in the mold temperature, T_1^*, is to be found.

We note from equation (39) that if T_1^* permits a power series expansion in powers of τ, the corresponding perturbation in shell thickness can be expanded in a series of odd half powers of τ. The simplest case of the inverse problem therefore occurs if the given function $H_1(\tau)$ has such an expansion, i.e. if

$$H_1(\tau)=\sum_{n=0}^{\infty}\varepsilon_n\tau^{n+1/2}. \quad (40)$$

It then follows that the temperature field can again be described in the form of equation (32). One equation for the unknown coefficients A_n, B_n is obtained by eliminating H_1 between equations (39) and (40) and equating coefficients of powers of τ, to obtain

$$A_nW(n+1/2,1/2)+B_n\lambda W(n+1,3/2)=-\lambda\Lambda\,e^{\lambda^2}\sum_{j=0}^{n}\frac{\varepsilon_j}{(n-j)!}. \quad (41)$$

A second equation is obtained by eliminating $H_1(\tau)$, $T_1(\lambda,\tau)$ between equations (23), (32), and (40) giving

$$4A_n\lambda nW(n+1/2,3/2)+B_nW(n,1/2)=2\Lambda\,e^{\lambda^2}\sum_{j=0}^{n}\frac{(1/2+j+\lambda^2)\varepsilon_j}{(n-j)!}. \quad (42)$$

The solution of equations (41) and (42) is routine, after which the unknown perturbation in mold temperature T_1^* can be recovered by substituting for A_n, B_n into equation (32) and setting $Y=0$.

Other forms of the function $H_1(\tau)$ can be accommodated by taking different values of α in equation

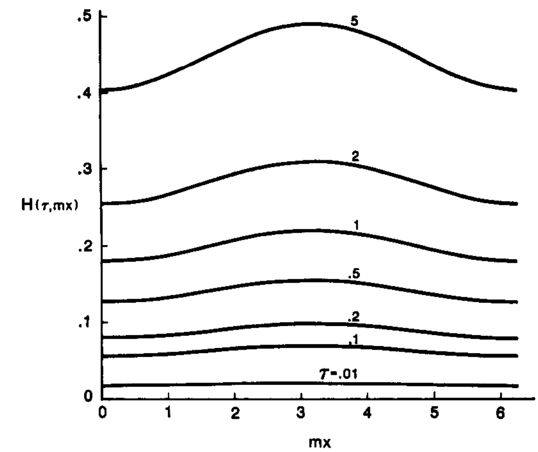

FIG. 2. Growth of the solidified shell with a sinusoidal variation in mold temperature: $F_1/\Delta T_0 = 0.2$, $\lambda = 0.1$.

(31). For example, if H_1 has a Taylor series expansion near $\tau = 0$, the odd half powers of τ are required, leading to the representation

$$T_1(Y,\tau) = e^{-(\tau+Y^2)} \sum_{n=0}^{\infty} \tau^{n+1/2}[D_n\Phi(1+n, 1/2; Y^2) + E_n Y\Phi(3/2+n, 3/2; Y^2)] \quad (43)$$

from which

$$H_1(\tau) = \sum_{n=1}^{\infty} \varepsilon_n \tau^n = -\frac{e^{-(\tau+\lambda^2)}}{\Lambda\lambda} \sum_{n=1}^{\infty} \tau^n [D_n\Phi(n, 1/2; \lambda^2) + E_n\lambda\Phi(n+1/2, 3/2; \lambda^2)]. \quad (44)$$

As before, equation (44) defines one equation for the unknown constants, D_n, E_n and a second equation is obtained from equations (23), (43), and (44). We note that the series in equation (44) cannot contain a constant term, since the physical statement of the problem demands $H_1 \ll H_0$ for all τ (see Fig. 1) and $H_0 = 0$ at $\tau = 0$. In equation (43), the second hypergeometric function can be expressed in terms of an Hermite polynomial.

More generally, the temperature distribution might be represented in integral form, treating α ($\equiv 4s$) as a parameter, i.e.

$$T_1(Y,\tau) = e^{-(\tau+Y^2)} \int_0^{\infty} \tau^s [A(s)\Phi(1/2+s, 1/2; Y^2) + B(s)Y\Phi(1+s, 3/2; Y^2)]\,ds \quad (45)$$

in which case the boundary conditions for either the direct or the inverse problem will lead to a pair of integral equations for the unknown functions $A(s)$, $B(s)$.

3. DISCUSSIONS AND RESULTS

Figure 2 shows the position of the solid/melt interface at various values of the dimensionless time, τ, for the case where $\lambda = 0.1$ and the perturbation in mold temperature is constant and equal to 20% of ΔT_0 (i.e. $p = 0$, $F_1/\Delta T_0 = 0.2$).

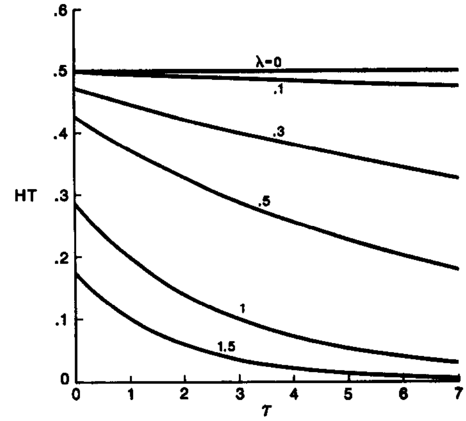

FIG. 3. Variation of amplitude of thickness variation with time for a constant mold temperature perturbation.

As long as the sinusoidal perturbations in dimensionless shell thickness, H_1, and mold temperature, T_1^*, are sufficiently small, they are linearly related and hence a more general presentation of results can be achieved by plotting the ratio

$$HT = \frac{-H_1(\tau)}{H_0(\tau)} \frac{\Delta T_0}{F_1} \quad (46)$$

as a function of τ. It is convenient to introduce a negative sign in definition (46), since a perturbation corresponding to a local increase in mold temperature will produce a local *decrease* in solidified shell thickness.

Results for various values of λ are given in Fig. 3. Notice that in the limit, $\lambda \to 0$, HT becomes independent of τ, whereas at larger values of λ, HT decays with τ, indicating that the sinusoidal component in the shell thickness grows more slowly than the mean thickness. The value of HT as $\tau \to 0$ depends only upon the constants A_0, B_0 and is shown as a function of λ in Fig. 4.

As might be anticipated, an exponential variation in the sinusoidal component of the mold temperature

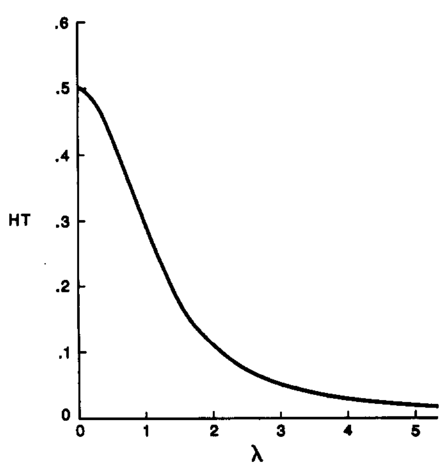

FIG. 4. Growth rate of thickness perturbation at small values of time.

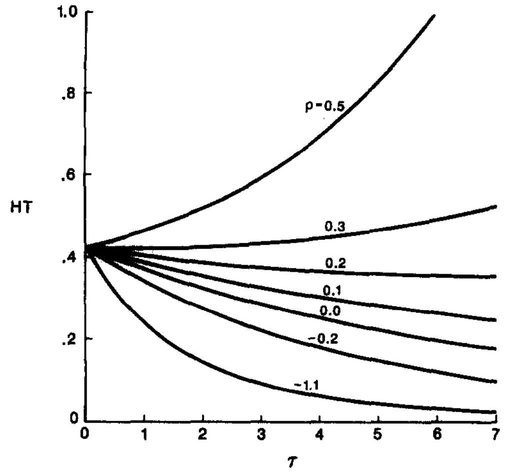

FIG. 5. Variation of shell thickness for an exponentially varying mold temperature perturbation: $\lambda = 0.5$.

has a major effect on the growth of the corresponding term in the shell thickness, as shown in Fig. 5 for $\lambda = 0.5$. In cases where $p > 0$, the solution will of course only satisfy the smallness criterion (see equation (9)) for a limited period of time which depends upon the magnitude of the initial perturbation.

Finally, we give some results for the inverse problem in which the solid/melt boundary is prescribed to grow according to the expression $H_1(\tau) = \varepsilon_0\sqrt{\tau}$. The required perturbation in mold temperature as a function of time is shown in Fig. 6 for various values of λ. In general, the variation in mold temperature must increase with time to preserve a constant growth in H_1, as might be expected from the results shown in Fig. 3. We note that ε_0 must be sufficiently small to ensure that $|T_1^*|/\Delta T_0 \ll 1$ for all τ for the linear perturbation solution to be appropriate.

4. CONCLUSIONS

The heat conduction problem has been investigated in which a one-dimensional solidification process is perturbed by the superposition of a small spatial sinusoidal variation in temperature parallel to the plane of the mold surface. In particular, solutions have been obtained for the temperature field and the advance of the solid/melt boundary for the case where the mold temperature varies sinusoidally with position. We also briefly discussed the inverse problem, in which the solid/melt boundary has a prescribed sinusoidal shape as a function of time and the corresponding mold temperature is to be found.

The results of the present analysis can be used in determining the residual thermoelastic stress in a casting with a non-uniform mold temperature. This problem is the subject of an ongoing investigation.

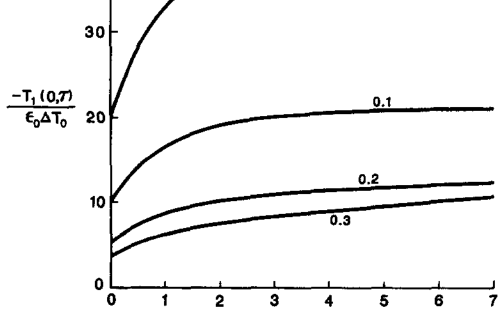

FIG. 6. The inverse problem. Mold temperature perturbation required to maintain a thickness perturbation, $H_1(\tau) = \varepsilon_0\sqrt{\tau}$.

Acknowledgements—The authors are pleased to acknowledge the support of the National Science Foundation under contract number MSM-8711557 and from the Aluminum Company of America.

REFERENCES

1. J. Stefan, Uber die Theorie der Eisbildung, Insbesondere uber die Eisbildung in Polarmeere, *Ann. Phys. Chem.* **42**, 269–286 (1891).
2. G. S. H. Lock, On the use of asymptotic solutions to plane ice–water problems, *J. Glaciology* **8**, 285–300 (1969).
3. R. H. Tien and O. Richmond, Theory of maximum tensile stresses in the solidifying shell of a constrained rectangular casting, *J. Appl. Mech.* **49**, 481–486 (1982).
4. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd Edn), p. 285. Oxford University Press, London (1959).
5. T. R. Goodman, The heat balance integral and its application to problems involving a change of phase, *Trans. ASME* **80**, 335–342 (1958).
6. L. T. Yeh and B. T. F. Chung, Solidification and melting of material subjected to convection and radiation, *J. Spacecr. Rockets* **12**, 329–334 (1975).
7. B. A. Boley, The embedding technique in melting and solidification problems. In *Moving Boundary Problems in Heat Flow and Diffusion* (Edited by J. R. Ockendon and W. R. Hodgkins), pp. 150–172. Oxford University Press, London (1974).
8. A. Lazaridis, A numerical solution of the multidimensional solidification (or melting) problem, *Int. J. Heat Mass Transfer* **13**, 1459–1477 (1970).
9. R. D. Pehlke, Computer simulation of solidification of a steel casting, *Proc. 1st Int. Steel Foundry Congress*, pp. 337–344. Steel Founder's Society of America (1985).
10. G. Comini, S. del Guidice, R. Lewis and O. Zienkiewicz, Finite element solution of non-linear heat conduction problems with special reference to phase change, *Int. J. Numer. Meth. Engng* **8**, 613–624 (1974).
11. W. Rolph III and K.-J. Bathe, An efficient algorithm for analysis of nonlinear heat transfer with phase changes, *Int. J. Numer. Meth. Engng* **18**, 119–134 (1982).
12. P. J. Wray, Non-uniform growth of a plate on a chilled surface, presented at the Annual Meeting of A.I.M.E., Atlanta, Georgia, 6–10 March (1977).
13. O. Richmond and N. C. Huang, Interface instability during unidirectional solidification of a pure metal, *Proc. 6th Canadian Congress of Applied Mechanics*, Vancouver, pp. 453–454 (1977).
14. J. R. Barber, Stability of thermoelastic contact, *Proc. Int. Conf. on Tribology*, Vol. 2, pp. 981–986. Institution of Mechanical Engineers, London (1987).

15. R. I. Pedroso and G. A. Domoto, Exact solution by perturbation method for planar solidification of a saturated liquid with convection at the wall, *Int. J. Heat Mass Transfer* **13**, 1459–1477 (1970).
16. W. W. Mullins and R. F. Sekerka, Stability of a planar interface during solidification of a dilute binary alloy, *J. Appl. Phys.* **35**, 444–451 (1964).
17. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Dover, New York (1965).

SOLUTIONS DE PERTURBATION SINUSOIDALE POUR LA SOLIDIFICATION PLANE

Résumé—Une méthode de perturbation linéaire est utilisée pour résoudre le problème de conduction thermique bidimensionnelle dans lequel un liquide, initialement à la température de fusion, se solidifie sur un plan dont la température est approximativement uniforme, mais qui contient une petite perturbation sinusoïdale dans une dimension d'espace. Les résultats sont obtenus pour une perturbation sinusoïdale à la frontière bain/solide en fonction du temps et pour la distribution de température dans le solide. Ces résultats sont exprimés par une série de fonctions hypergéométriques confluentes qui possède une bonne convergence pour des valeurs réelles des propriétés des matériaux et de températures. On discute aussi brièvement le problème inverse dans lequel la frontière bain/solide est prescrite et sa température doit être déterminée.

SINUSFÖRMIGE STÖRUNGSANSÄTZE FÜR DIE EBENE ERSTARRUNG

Zusammenfassung—Es wird eine lineare Störungsmethode benutzt, um das zweidimensionale Wärmeleitungsproblem zu lösen, bei dem eine Flüssigkeit, die sich anfänglich auf Schmelztemperatur befindet, erstarrt. Dies geschieht durch Wärmeübertragung an eine ebene Form, deren Temperatur näherungsweise einheitlich ist, aber in einer Raumrichtung eine leichte sinusförmige Schwankung aufweist. Daraus folgt eine sinusförmige zeitabhängige Störung der Erstarrungsfront. Ergebnisse hierfür und für die Temperaturverteilung im bereits verfestigten Mantel werden mitgeteilt. Diese Ergebnisse werden mit Hilfe einer Reihe konfluenter hypergeometrischer Funktionen ausgedrückt, die eine gute Übereinstimmung für praktische Werte der Materialeigenschaften und der Temperatur zeigen. Das inverse Problem, bei dem die Erstarrungsfront vorgeschrieben und die Temperatur der Form zu bestimmen ist, wird ebenfalls diskutiert.

РЕШЕНИЕ ДЛЯ ЗАТВЕРДЕВАНИЯ С ПЛОСКОЙ ГРАНИЦЕЙ ПРИ СИНУСИДАЛЬНОМ ВОЗМУЩЕНИИ

Аннотация—Методом возмущений решается двумерная задача теплопроводности, в которой жидкость с первоначальной температурой плавления затвердевает за счет теплопереноса к плоской прессформе, температура которой почти постоянна, но имеет малое синусоидальное возмущение по одной координате. Получены результаты для возникающего синусоидального возмущения как функции времени на границе расплава и твердого вещества и для изменения температуры во всей твердой фазе. Эти результаты представлены в виде ряда по гипергеометрическим функциям, хорошо сходящегося для реальных значений характеристик вещества и температуры. Кратко обсуждается обратная задача, в которой задана граница между расплавом и твердым веществом, а температура прессформы подлежит определению.